1. Laplace equation

We call $\Delta u = 0$ the Laplace equation, and we call its solution u a harmonic function. Given a smooth function $f(x)$, we call $\Delta u = f$ the Poisson's equation.

Let Ω denote a bounded open subset in $\mathbb{R}^2$ with a smooth boundary curve $\partial\Omega$.

2. Harmonic function

Let us investigate several properties of harmonic functions.

Theorem 1 (Uniqueness)**.** *Given a smooth function $g : \partial\Omega \to \mathbb{R}$, there exists a unique smooth harmonic function $u : \Omega \to \mathbb{R}$ satisfying $u = g$ on $\partial\Omega$.*

Proof. Suppose that we have two solutions u, v, and define $w = u - v$. Then, we have $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. Thus,

$$\int_\Omega |Dw|^2 dx = \int_{\partial\Omega} w\partial_\nu w dx - \int_\Omega w\Delta w dx = 0.$$

Therefore, w is a constant, and thus $w = 0$ by the boundary condition. □

Theorem 2 (Mean value property)**.** *A harmonic function u satisfies*

$$u(0) = \frac{1}{\pi r^2}\int_{B_r(0)} u(x)dx = \frac{1}{2\pi r}\int_{\partial B_r(0)} u(x)ds.$$

Proof. We begin by defining

$$h(r) = \frac{1}{2\pi r}\int_{\partial B_r(0)} u(x)ds$$

for $r > 0$. Then, we have

$$h(r) = \frac{1}{2\pi r}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)rd\theta = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)d\theta$$

Hence,

$$\begin{aligned} h'(r) &= \frac{1}{2\pi}\int_0^{2\pi} \langle \nabla u(r\cos\theta, r\sin\theta), (\cos\theta, \sin\theta)\rangle d\theta \\ &= \frac{1}{2\pi}\int_{\partial B_r(0)} \langle \nabla u(r\cos\theta, r\sin\theta), (\cos\theta, \sin\theta)\rangle \frac{1}{r} ds \\ &= \frac{1}{2\pi r}\int_{\partial B_r(0)} \partial_\nu u ds = \frac{1}{2\pi r}\int_{B_r(0)} \Delta u dx = 0. \end{aligned}$$

Since $h(r)$ is a constant and $\lim_{r\to 0} u(0)$, we have $u(0) = h(r)$ the second formula. The first one can be obtained as follows.

$$\frac{1}{\pi r^2}\int_{B_r(0)} u(x)dx = \frac{1}{\pi r^2}\int_0^t \int_{\partial B_t(0)} u(x)dsdt = \frac{1}{\pi r^2}\int_0^r 2\pi t\, u(0)dt = u(0).$$

□

We call a function u subharmonic [resp.superharmonic] if it satisfies $\Delta u \geqslant 0$ [resp. $\Delta u \geqslant 0$].

Proposition 3. *A subharmonic function satisfies*

$$u(0) \leqslant \frac{1}{\pi r^2}\int_{B_r(0)} u(x)dx,$$
$$u(0) \leqslant \frac{1}{2\pi r}\int_{\partial B_r(0)} u(x)ds.$$

A superharmonic function satisfies

$$u(0) \geqslant \frac{1}{\pi r^2}\int_{B_r(0)} u(x)dx,$$
$$u(0) \geqslant \frac{1}{2\pi r}\int_{\partial B_r(0)} u(x)ds.$$

Proof. Remind that $\Delta u \geqslant 0$ implies $h'(r) \geqslant 0$ in the proof of MVP. One can easily modify the proof above. □

3. Maximum principle

We establish the maximum principle for a general class of linear elliptic PDEs. A simple proof of the maximum principle for harmonic functions is provided in the textbook chapter 3.3.

In this subsection, we consider $a_{ij}(x)$, $b_i(x)$, $c(x)$ are smooth functions defined on $\overline{\Omega}$ satisfying

$$a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \tag{1}$$

for some constant $\lambda > 0$, where $\xi \in \mathbb{R}^n$. In addition, $a_{ij}(x)$ is a symmetric matrix at each x, namely $a_{ij}(x) = a_{ji}(x)$. We define a linear differential operator L by

$$Lu = a_{ij}\partial_{ij}u + b_i\partial_i u + cu. \tag{2}$$

We recall the eigenvalue decomposition from Linear algebra. For each x there exists real numbers $\lambda_1(x), \cdots, \lambda_n(x)$ and unit vectors $\vec{q}_1(x), \cdots, \vec{q}_n(x) \in \mathbb{R}$ such that $\lambda_i \geqslant \lambda$, $\langle \vec{q}_i, \vec{q}_j \rangle = 0$ for $i \neq j$, and

$$a_{ij} = \sum_{k=1}^{n} \lambda_k q_i^k q_j^k, \tag{3}$$

where $\vec{q}_k = (q_1^k, \cdots, q_n^k)$.

Lemma 4. *Suppose that $Lu > 0$ and $c(x) \leqslant 0$ hold in Ω. Then, the smooth subsolution u satisfies*

$$\max_{\overline{\Omega}} u \leqslant \max_{\partial\Omega} u_+,$$

where $u_+ = \max\{0, u\}$.

Proof. Assume that u attains its maximum at an interior point $x_0 \in \Omega$ and $u(x_0) > 0$. Then, at x_0 we have

$$0 < Lu = a_{ij}u_{ij} + b_i u_i + cu \leqslant a_{ij}u_{ij},$$

by $u_i(x_0) = 0$, $c \leqslant 0$, and $u(x_0) > 0$. In addition, by (3).

$$0 < a_{ij}u_{ij} = \sum_{i,j,k=1}^{n} \lambda_k (q_i^k q_j^k u_{ij}).$$

However, a function $h(t) = u(x_0 + t\vec{q}_k(x_0))$ attains its maximum at $t = 0$. Hence

$$0 \geqslant h''(0) = q_i^k q_j^k u_{ij}(x_0), \tag{4}$$

namely $0 < a_{ij}u_{ij} \leqslant 0$. Contradiction. □

Theorem 5 (Weak maximum principle)**.** *Suppose that $Lu \geqslant 0$ and $c(x) \leqslant 0$ hold in Ω. Then, the smooth subsolution u satisfies*

$$\max_{\overline{\Omega}} u \leqslant \max_{\partial\Omega} u_+,$$

where $u_+ = \max\{0, u\}$.

Proof. We define $w = u + \epsilon e^{-\alpha x_1}$ for $\epsilon > 0$ and $\alpha \in \mathbb{R}$. Then,

$$Lw = Lu + \epsilon L e^{-\alpha x_1} \geqslant \epsilon L e^{-\alpha x_1}.$$

Moreover,

$$Le^{-\alpha x_1} = e^{-\alpha x_1}\Big[\alpha^2 a_{11} + \alpha b_1 + c\Big] \geqslant e^{-\alpha x_1}\Big[\lambda\alpha^2 + \alpha b_1 + c\Big].$$

Since $\lambda > 0$ and b_i, c are bounded, we can choose sufficiently large α depending on λ, b_i, c such that $Le^{-\alpha x_1} > 0$. Then, we have $Lw > 0$. Thus, Lemma 4 yields

$$\max_{\overline{\Omega}} u \leqslant \max_{\overline{\Omega}} w \leqslant \max_{\partial\Omega} w_+ \leqslant \max_{\partial\Omega} u_+ + \epsilon \max_{\partial\Omega} e^{-\alpha x_1}.$$

Passing $\epsilon \to 0$ yields the desired result. □

Lemma 6 (Hopf)**.** *Suppose that $Lu \geqslant 0$ and $c(x) \leqslant 0$ hold in an open ball B. Moreover, there exists a boundary point $x_0 \in \partial B$ satisfying $u(x_0) \geqslant 0$ and $u(x_0) > u(x)$ for $x \in B$. Then, the following holds*

$$\partial_\nu u(x_0) > 0.$$

Proof. By translating the ball B, we may assume $x_0 \in \partial B_r(0)$ and $B_r(0) \subset B$. Next, we define

$$\Omega = B_r(0) \cap B_{r/2}(x_0).$$

We consider a function $v = u + \epsilon h$, where $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$. Then, in Ω

$$\begin{aligned} Lh &= e^{-\alpha|x|^2}\big[4\alpha^2 a_{ij}x_i x_j - 2\alpha \sum_{i=1} \alpha_{ii} + 2b_i x_i + c\big] - ce^{-\alpha r^2} \\ &\geqslant e^{-\alpha|x|^2}\big[4\alpha^2\lambda|x|^2 - 2\alpha \sum_{i=1}^{n} \big(a_{ii} + |b_i||x|\big) + c\big]. \end{aligned}$$

Since $|x|^2 \geqslant \frac{r^2}{4}$ and $|x| \leqslant r$ holds in Ω, we have $Lh > 0$ by choosing a sufficiently large α. Namely, we have $Lv = Lu + \epsilon Lh > 0$, and thus Lemma 4 yields

$$\max_{\overline{\Omega}} v \leqslant \max_{\partial\Omega} v_+. \tag{5}$$

Now, we claim that there exists a small enough ϵ such that $v(x_0) = \max_{\partial\Omega} v_+$. First of all, on the portion $\partial B_r(0) \cap B_{r/2}(x_0) \subset \partial\Omega$ we have $h = 0$. Hence, $v_+ = u_+ \leqslant u(x_0) = v(x_0)$. Next, the other portion $B_r(0) \cap \partial B_{r/2}(x_0) \subset \partial\Omega$ is a compact subset of the open set B, where $u(x) < u(x_0)$ holds. Hence, there exists a small $\delta > 0$ such that $u(x) \leqslant u(x_0) - \delta$ holds on $B_r(0) \cap \partial B_{r/2}(x_0)$. Since h is bounded over $\overline{\Omega}$, we can choose small enough ϵ such that $\epsilon h \leqslant \delta$. Then, $v = u + \epsilon h \leqslant u + \delta \leqslant u(x_0) = v(x_0)$ holds on $B_r(0) \cap \partial B_{r/2}(x_0)$.

In conclusion, we have $v(x_0) = \max_{\partial\Omega} v_+ = \max_{\overline{\Omega}} v$. Thus,

$$\partial_\nu u(x_0) = \partial_\nu v(x_0) - \epsilon \partial_\nu h(x_0) \geqslant -\epsilon \partial_\nu h(x_0) > 0. \tag{6}$$

□

Theorem 7 (Strong maximum principle)**.** *Suppose that $Lu \geqslant 0$ and $c(x) \leqslant 0$ hold in Ω. Then, the smooth subsolution u is a constant in $\overline{\Omega}$ or*

$$u(x) < \max_{\partial\Omega} u_+,$$

holds for $x \in \Omega$.

Proof. Assume that u attains its maximum at an interior point $x_0 \in \Omega$ and $u(x_0) = M \geqslant 0$. We define a set $\Sigma = \{x \in \overline{\Omega} : u(x) = M\}$. Since u is a continuous function, Σ is a closed set. Towards contradiction, we assume Ω is not contained in Σ. Then, there exists a point $y_0 \in \Omega \backslash \Sigma$ such that $d(y_0, \partial\Omega) > d(y_0, \Sigma)$, where $d(y_0, A)$ denotes the distance from y_0 to the set A. There exists a small $r > 0$ such that $B_r(y_0) \subset \Omega \backslash \Sigma$, because $\Omega \backslash \Sigma$ is an open set. Next, we define R by

$$R = \sup\{r : B_r(y_0) \cap \Omega \backslash \Sigma\}.$$

Then, there exists a point $z_0 \in \Sigma \cap \partial B_R(y_0)$ and $z_0 \in \Omega$. Since $u(z_0) = \max u$ and $z_0 \in \Omega$, we have $Du(z_0) = 0$. However, by the Hopf's Lemma, we have $\partial_\nu u(z_0) > 0$ where ν is the outward pointing direction of $\partial B_R(y_0)$. Contradiction. □